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'Lazy' quantum ensembles

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Abstract

We compare different strategies aimed to prepare an ensemble with a given density matrix ρ. Preparing the ensemble of eigenstates of ρ with appropriate probabilities can be treated as 'generous' strategy: it provides maximal accessible information about the state. Another extremity is the so-called 'Scrooge' ensemble, which is mostly stingy in sharing the information. We introduce 'lazy' ensembles which require minimal effort to prepare the density matrix by selecting pure states with respect to completely random choice. We consider two parties, Alice and Bob, playing a kind of game. Bob wishes to guess which pure state is prepared by Alice. His null hypothesis, based on the lack of any information about Alice's intention, is that Alice prepares any pure state with equal probability. Then, the average quantum state measured by Bob turns out to be ρ, and he has to make a new hypothesis about Alice's intention solely based on the information that the observed density matrix is ρ. The arising 'lazy' ensemble is shown to be the alternative hypothesis which minimizes type I error.

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Introduction

Consider two parties, Alice and Bob, playing the following game. Alice prepares a pure quantum state according to certain random strategy and then sends it to Bob. Initially Bob possesses no information about Alice's strategy and thus assumes that Alice performs a completely random choice of pure state, we refer to this statement as a *null hypothesis*. In this case the average density matrix received by Bob would be proportional to identity.

Measuring the received states, Bob realizes that the average quantum state emitted by Alice is ρ. However, there are infinitely many ensembles which average to ρ, and Bob still cannot recover the strategy of Alice. Although Bob now possesses some information about

Alice's intentions: if the received density matrix ρ differs from identity, Bob has to make an alternative hypothesis. To specify such a hypothesis, some extra principles must be taken into account. These principles should capture the type of Alice's behaviour.

We might assume that the strategy of Alice is to prepare eigenstates of ρ with given probabilities, but this is just an assumption that Alice is 'generous' in providing the accessible information. Or, conversely, Alice might be stingy with the information and thus chooses pure states according to Scrooge distribution [1].

In our setting, Bob is reluctant to change his opinion and chooses among Alice's strategies (which average to ρ) the closest to his null hypothesis. By 'closest' we mean minimizing the Kullback–Leibler [4] distance between the distributions. This distance is the average likelihood ratio and is associated with the probability of type I error[3].

Another way for Bob's reasonings is to assume Alice to be lazy in efforts to prepare the ensemble. These efforts are quantified in terms of differential entropy. Remarkably, as we show in section 1, both approaches yield the same ensemble (4).

The resulting ensemble is a continuous distribution over the set of *all* one-dimensional projectors $|\varphi\rangle\langle\varphi|$ in the state space of the system. Such kind of ensembles were considered earlier in the literature. Best known is the uniform ensemble yielding completely mixed state, it was used for various purposes, for instance, to evaluate the entangling power of unitary transformations [5]. Another example is Scrooge ensemble. If Alice's choice would be this ensemble for a given density matrix $\rho = \sum \lambda_k |\mathbf{e}_k\rangle\langle\mathbf{e}_k|$:

$$\rho = \int \sqrt{\rho}|\varphi\rangle\langle\varphi|\sqrt{\rho}\,\mathrm{d}\varphi, \tag{1}$$

then the measurements carried out by Bob would give the least information about Alice's action—provided that the average state ρ prepared by Alice is known—in contrast with the quantum state estimation problem [6], where the state ρ is to be determined.

1. Differential entropy and the likelihood ratio

First we have to specify a yet vague notion of 'preparation efforts' for an ensemble. Following [2] we formulate it in thermodynamic terms, namely, we quantify these efforts by the difference between the entropy of uniform distribution (that is, our null hypothesis) and the entropy of the ensemble in question. The only obstacle may occur is to define this entropy, let us dwell on it in more detail.

The entropy of a finite distribution $\{p_i\}$ is given by the Shannon formula

$$S(\{p_i\}) = -\sum p_i \ln p_i.$$

This expression diverges for any continuous distribution: we approximate a continuous distribution $\mu(\varphi)$ by a discrete one $\{p_i\}$, calculate its Shannon entropy, but it tends to infinity as we refine the partition. However, we are always interested in the *difference* between the entropy of the uniform distribution and the distribution $\mu(\varphi)$ rather than the entropy itself. At each approximation step we calculate this difference, and the appropriate limit always exists. To show it (see [3] for details), make a partition of the probability space by N sets Δ_i having an equal uniform measure. Then the difference E_N between the entropies read

$$E_N = \ln N - \left(-\sum p_i \ln p_i\right),$$

[3] To make type I error means to accept the alternative hypothesis when the null hypothesis is still valid. An example of type I error is a wrong accusing sentence to an innocent person.

where $p_i = \int_{\Delta_i} p(\varphi)\,\mathrm{d}\varphi$. The limit expression $\lim_{N\to\infty} E_N$ is the differential entropy

$$S(\mu) = \int \mu(\varphi) \ln \mu(\varphi)\,\mathrm{d}\varphi. \tag{2}$$

This is equal to the Kullback–Leibler distance [4],

$$S(\mu\|\mu_0) = \int \mu(\varphi) \ln \frac{\mu(\varphi)}{\mu_0(\varphi)}\,\mathrm{d}\varphi,$$

between the distribution $\mu(\varphi)$ and the uniform distribution $\mu_0(\varphi)$ with constant density[4], normalize the counting measure dx on the probability space so that $\mu_0(\varphi) \equiv 1$. This distance is the average likelihood ratio, on which the choice of statistical hypothesis is based. Then, in order to minimize type I error we have to choose a hypothesis with the smallest average likelihood ratio.

2. ‘Lazy’ ensembles

The main problem reduces to the following. For given density matrix ρ find a continuous ensemble μ having minimal differential entropy (2):

$$S(\mu) \to \min, \qquad \int |\varphi\rangle\langle\varphi|\mu(\varphi)\,\mathrm{d}\varphi = \rho, \tag{3}$$

where φ labels *all*[5] one-dimensional projectors $|\varphi\rangle\langle\varphi|$ in the state space and dφ is the unitary invariant measure on pure states normalized to integrate to unity. When there is no constraints in (3), the answer is straightforward—the minimum (equal to zero) is attained on uniform distribution. To solve the problem with constraints, we use the Lagrange multiples method. The appropriate Lagrange function reads

$$\mathcal{L}(\mu) = S(\mu) - \mathrm{Tr}\,\Lambda\left(\int |\varphi\rangle\langle\varphi|\mu(\varphi)\,\mathrm{d}\varphi - \rho\right),$$

where the Lagrange multiple Λ is a matrix since the constraints in (3) are of matrix character. Substituting the expression (2) for $S(\mu)$ and making the derivative of $\mathcal{L}$ over μ zero, we get

$$\mu(\varphi) = \frac{\mathrm{e}^{-\mathrm{Tr}\,B|\varphi\rangle\langle\varphi|}}{Z(B)}, \tag{4}$$

where B is the optimal value of the Lagrange multiple Λ which we derive from the constraint (3) and the normalizing multiple

$$Z(B) = \int \mathrm{e}^{-\mathrm{Tr}\,B|\varphi\rangle\langle\varphi|}\,\mathrm{d}\varphi \tag{5}$$

is the partition function for (4). Substituting the resulting density (4) to expression (2) for differential entropy we get

$$S = \mathrm{Tr}\,B\rho - \ln Z, \tag{6}$$

in particular, if we fix the gauge condition $Z(B) = 1$, the above expression reduces to

$$S = \mathrm{Tr}\,B\rho \tag{7}$$

which gives the observable B the meaning of differential entropy itself.

4 Note that to define a distribution also means to specify the probability space, that is why these distributions different for different dimensions of the state space of the system in question.

5 Note that for $\varphi \neq \varphi'$ the appropriate projectors $|\varphi\rangle\langle\varphi|$ and $|\varphi'\rangle\langle\varphi'|$ are *not* orthogonal, therefore $|\varphi\rangle\langle\varphi|\mu(\varphi)$ is *not* a spectral measure.

3. Special case: qubit

In this case the state space has dimension 2. Write down the parameter B in the eigenbasis of the density matrix ρ in a suitable form

$$B = b \cdot \mathbb{I} + \begin{pmatrix} -\beta & 0 \\ 0 & +\beta \end{pmatrix}. \tag{8}$$

Then expression (12) for partition function reads:

$$Z = \mathrm{e}^{-b} \cdot \frac{\mathrm{e}^{\beta} - \mathrm{e}^{-x}}{2\beta} = \mathrm{e}^{-b} \cdot \frac{\sinh \beta}{\beta}. \tag{9}$$

Calculating the partial derivatives according to (14), we get the following expressions for the coefficients $\lambda_{1,2}$ of the density matrix,

$$\lambda_{1,2} = \frac{1}{2} \pm \frac{1}{2}\left(\coth \beta - \frac{1}{\beta}\right) = \frac{1}{2} \pm \delta, \tag{10}$$

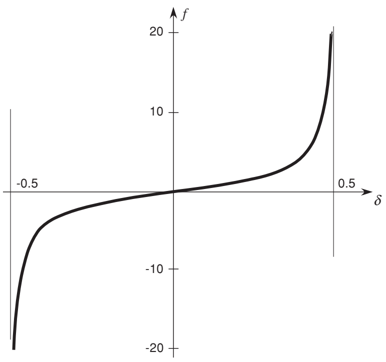
where

$$\delta = \frac{1}{2}\left(\coth \beta - \frac{1}{\beta}\right). \tag{11}$$

Denote by $f(\delta)$ the inverse to δ. Since the δ is odd and a monotone function of β, its inverse f exists and bears the same properties. Then the matrix B (8) is the following function of the density matrix:

$$B = b \cdot \mathbb{I} + \begin{pmatrix} f\left(\frac{\lambda_2 - \lambda_1}{2}\right) & 0 \\ 0 & f\left(\frac{\lambda_1 - \lambda_2}{2}\right) \end{pmatrix} = b \cdot \mathbb{I} + f\left(\frac{\mathbb{I}}{2} - \rho\right).$$

Since expression (11) for ρ is the independence of the choice of b, in two-dimensional case both matrices B and ρ are defined by their mean deviation values β and δ, respectively. So, the essential dependence of the matrix 'temperature' parameter B from the density matrix ρ is completely captured by the function f. Its graph looks as follows:



4. Explicit expressions for higher dimensions

Now let the space has arbitrary finite dimension n. First evaluate the partition function (5) in the eigenbasis of B. Since the quadratic form in the exponent does not depend on phases, the integration can be carried out over the probability simplex [8] and $Z(B)$ reads

$$Z(B) = -(n-1)! \sum_{k=1}^{n} \frac{\mathrm{e}^{-b_k}}{\prod_{j \neq k}(b_k - b_j)}, \tag{12}$$

where b_k are the eigenvalues of B. If two or more of them are equal, the appropriate expression is obtained as a limit starting with unequal eigenvalues. The detailed derivation of this formula can be found in the appendix. To write down the expression for the eigenvalues λ_s of the density matrix ρ via B we could evaluate the integrals

$$\lambda_s = \langle \mathbf{e}_s | \int |\varphi\rangle\langle\varphi| \mu(\varphi)\, \mathrm{d}\varphi | \mathbf{e}_s \rangle$$

in the eigenbasis of ρ. Although, like in thermodynamics, we have

$$\rho = \frac{\partial \ln Z}{\partial B}, \tag{13}$$

which gives the explicit expression for the eigenvalues of the density matrix ρ:

$$\lambda_s = -\frac{\frac{\mathrm{e}^{-b_s}}{\prod_{\substack{j=1\\ j\neq s}}^n (b_s-b_j)} + \sum_{\substack{k=1\\ k\neq s}}^n \frac{1}{b_s-b_k} \cdot \left(\frac{\mathrm{e}^{-b_s}}{\prod_{\substack{j=1\\ j\neq s}}^n (b_s-b_j)} + \frac{\mathrm{e}^{-b_k}}{\prod_{\substack{j=1\\ j\neq k}}^n (b_k-b_j)} \right)}{\sum_{k=1}^n \frac{\mathrm{e}^{-b_k}}{\prod_{\substack{j=1\\ j\neq k}}^n (b_k-b_j)}}, \tag{14}$$

from which we see that the resulting density matrix ρ remains unchanged when we add a constant to all b_k-s. That means that the matrix 'temperature' parameter B for the lazy ensemble is defined up to an additive constant (in contrast with classical thermodynamics).

Like in [1], expression (12) for the partition function can be given the following integral form

$$Z(B) = -\frac{(n-1)!}{2\pi \mathrm{i}} \oint \frac{\mathrm{e}^{-z}\, \mathrm{d}z}{\det(B - z\mathbb{I})}, \tag{15}$$

where the contour encloses all eigenvalues of B.

So, given a lazy ensemble (4) with the parameter B, we have written down expression (13) for its average density matrix. This expression is well defined for any matrix B. The existence problem remains: given a density matrix ρ, is there a lazy ensemble with appropriate parameter B which averages to ρ? Similar question—the existence of temperature function—arises in thermodynamics. The idea to solve it is the following [3]: we consider the n-dimensional CDF (cumulative density function) of the measure μ and study the asymptotics of its Laplace transform. As a result, B exists for any *full-range* density matrix ρ.

The question arises how, given the spectral decomposition of a density operator, to calculate the appropriate matrix parameter B. This is the inverse expression to (14). Unlike (14), the dependence of the spectrum of B from that of ρ is not an elementary function. However, if we fix the condition $Z(B) = 1$, the correspondence $\{\lambda_j\}_{j=1}^n \leftrightarrow \{b_j\}_{j=1}^n$ is one-to-one.

5. Lazy ensembles are equilibrium

Like Gibbs ensembles in thermodynamics, the lazy ensembles are *equilibrium*, namely, the introduced parameters B possess the equalizing property. To show it, first introduce the notion of conditional ensemble. In terms of game played by Alice and Bob, this means that Bob measures a fixed observable H upon the particles emitted by Alice. Again, he has the uniform distribution as null hypothesis, but the constraint in (3) is of scalar rather than of matrix character. Solving the appropriate variational problem,

$$S(\mu) \to \min, \qquad \int \mathrm{Tr}\, H|\varphi\rangle\langle\varphi|\mu(\varphi)\, \mathrm{d}\varphi = \mathrm{Tr}\, H\rho,$$

we obtain

$$\mu_H(\varphi) = \frac{e^{-\beta \operatorname{Tr} H|\varphi\rangle\langle\varphi|}}{Z_H(\beta)}, \tag{16}$$

this ensemble is *conditional* with respect to given observable H.

Consider two quantum systems with state spaces $\mathcal{H}$ and $\mathcal{H}'$, respectively. Let their states initially be ρ and ρ'. Then, since we consider a non-interacting coupling of the systems, the joint density matrix is $\rho \otimes \rho'$ in the tensor product space $\mathcal{H} \otimes \mathcal{H}'$. Let us measure the sum of values of the observables H and H', that is, introduce the observable $\mathbf{H} = H \otimes \mathbb{I}' + \mathbb{I} \otimes H'$. The conditional optimal ensemble of *separable* states with respect to the observable $\mathbf{H}$ is the following distribution:

$$\mu_{\mathbf{H}}(\varphi \otimes \varphi') = \frac{\exp[-\beta_{\mathbf{H}} \operatorname{Tr} \mathbf{H}|\varphi \otimes \varphi'\rangle\langle\varphi \otimes \varphi'|]}{Z_{\mathbf{H}}(\beta_H)}.$$

Like in classical thermodynamics, the partition function of the joint system is the product of subsystems' partition functions:

$$\begin{aligned} Z_{\mathbf{H}}(\tau) &= \iint \exp(-\tau \operatorname{Tr} \mathbf{H}|\psi \otimes \psi'\rangle\langle\psi \otimes \psi'|)\, d\varphi\, d\varphi' \\ &= \iint \exp(-\tau(\operatorname{Tr} H|\psi\rangle\langle\psi| + \operatorname{Tr} H'|\psi'\rangle\langle\psi'|))\, d\varphi\, d\varphi' = Z_H(\tau) \cdot Z_{H'}(\tau), \end{aligned}$$

therefore the equalizing property holds

$$\text{If} \quad \beta_H \leqslant \beta_{H'} \qquad \text{then} \qquad \beta_H \leqslant \beta_{\mathbf{H}} \leqslant \beta_{H'}, \tag{17}$$

which means that the conditional lazy ensembles are equilibrium.

6. Concluding remarks

Continuous ensembles of pure states proved their relevance in various aspects of quantum mechanics. From the theoretical perspective, they provide the limit cases on which numerical characteristics of density matrices are attained, for instance, the minimal value of accessible information about the state is attained on 'Scrooge' ensemble which is a continuous distribution [1]. Furthermore, we claim that they are relevant from the operationalistic point of view. Even if we are speaking of preparing discrete ensembles, we must also have in mind that their are unavoidably smeared by various noises and, strictly speaking, we have to deal with continuous distributions.

We use the techniques of continuous ensembles to carry out statistical inference in quantum realm according to the standard scheme: we have an *a priori* hypothesis (we necessarily need it, otherwise there is no way to make any inference [8]), then we obtain some information about the system and have to shift to a new hypothesis.

In our case the null hypothesis is the assumption that any pure state is emitted with equal probability. Then the information is obtained that the average density matrix of the state is ρ. We show how, starting from the 'minimal effort' assumption, to guess the strategy of the preparation of the pure states. As a result, we obtain so-called 'lazy' ensembles.

These ensembles are also proved to provide the minimal deviation from the null hypothesis. They are described by exponential distributions (4) of pure states averaging to a given density matrix ρ:

$$\rho = \int \frac{e^{-\langle\varphi|B|\varphi\rangle}}{Z(B)}\, d\varphi,$$

where the matrix parameter B is shown to possess the equalizing property (like temperature in classical thermodynamics). Although we may not treat it as a fully-fledged temperature, for instance, in contrast with classical thermodynamics, it is ambiguously defined up to an arbitrary additive constant. According to formula (6), we can so choose the additive gauge for B that $\ln Z$ will vanish and the mean value $\operatorname{Tr} B\rho$ will be equal to the differential entropy of the ensemble, so we may call this matrix parameter B 'differential entropy observable'.

The techniques we introduce differ crucially from quantum state estimation (see [6] for details). The main difference is that the state, which Alice prepares, is *known*, it is ρ. What we estimate, is the choice of Alice which strategy to follow.

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Appendix. The derivation of the explicit expression for the partition function

To prove formula (12) for the partition function, we do it recursively. Start from expression (5) in the eigenbasis of the operator $B = \sum b_k|\mathbf{e}_k\rangle\langle\mathbf{e}_k|$, where $t_k = |\langle\varphi|.\mathbf{e}_k\rangle|^2$:

$$\frac{1}{(n-1)!}\cdot Z_n(b_1,\ldots,b_n) = \frac{\mathrm{e}^{-b_n}}{\prod_{k=1}^{n-1}(b_k-b_n)}\cdot\int_{\sum_{k=1}^{n-1}t_k\leqslant 1}\prod_{k=1}^{n-1}\mathrm{d}\,\mathrm{e}^{-(b_k-b_n)t_k}.$$

Split out the variable t_{n-1} getting

$$\begin{aligned}
\frac{1}{(n-1)!}\cdot Z_n(b_1,\ldots,b_n) &= \frac{\mathrm{e}^{-b_n}}{\prod_{k=1}^{n-1}(b_k-b_n)}\cdot\int_{\sum_{k=1}^{n-2}t_k\leqslant 1}\prod_{k=1}^{n-2}\mathrm{d}\,\mathrm{e}^{-(b_k-b_n)t_k}\left(\mathrm{e}^{-(b_{n-1}-b_n)}\right)\Big|_{t=0}^{1-\sum_{k=1}^{n-2}t_k}\\
&= \frac{\mathrm{e}^{-b_n}}{\prod_{k=1}^{n-1}(b_k-b_n)}\cdot\int_{\sum_{k=1}^{n-2}t_k\leqslant 1}\prod_{k=1}^{n-2}\mathrm{d}\,\mathrm{e}^{-(b_k-b_n)t_k}\\
&\quad\cdot\left(\exp\left(-(b_{n-1}-b_n)\left(1-\sum_{k=1}^{n-2}t_k\right)\right)-1\right)\\
&= \frac{\mathrm{e}^{-b_n}}{\prod_{k=1}^{n-1}(b_k-b_n)}\cdot\int_{\sum_{k=1}^{n-2}t_k\leqslant 1}\left(\prod_{k=1}^{n-2}\mathrm{d}\,\mathrm{e}^{-(b_k-b_n)t_k}\right)\\
&\quad\times\exp\left(-(b_{n-1}-b_n)\left(1-\sum_{k=1}^{n-2}t_k\right)\right)\\
&\quad-\frac{\mathrm{e}^{-b_n}}{\prod_{k=1}^{n-1}(b_k-b_n)}\cdot\int_{\sum_{k=1}^{n-2}t_k\leqslant 1}\prod_{k=1}^{n-2}\mathrm{d}\,\mathrm{e}^{-(b_k-b_n)t_k}\\
&= \frac{\mathrm{e}^{-b_{n-1}}}{b_{n-1}-b_n}\cdot\int_{\sum_{k=1}^{n-2}t_k\leqslant 1}\exp\left(-\sum_{k=1}^{n-2}(b_k-b_n)t_k\right)\left(\prod_{k=1}^{n-2}\mathrm{d}t_k\right)\\
&\quad\times\exp\left((b_{n-1}-b_n)\sum_{k=1}^{n-2}t_k\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{b_{n-1}-b_n}\frac{\mathrm{e}^{-b_n}}{\prod_{k=1}^{n-2}(b_k-b_n)}\cdot\int_{\sum_{k=1}^{n-2}t_k\leqslant 1}\prod_{k=1}^{n-2}\mathrm{d}\,\mathrm{e}^{-(b_k-b_n)t_k}\\
&= \frac{\mathrm{e}^{-b_{n-1}}}{b_{n-1}-b_n}\cdot\int_{\sum_{k=1}^{n-2}t_k\leqslant 1}\exp\left(-\sum_{k=1}^{n-2}(b_k-b_{n-1})t_k\right)\\
&\times\prod_{k=1}^{n-2}\mathrm{d}t_k - \frac{1}{(n-2)!}\cdot\frac{Z_{n-1}(b_1,\ldots,b_{n-2},b_n)}{b_{n-1}-b_n}\\
&= \frac{1}{(n-2)!}\cdot\frac{Z_{n-1}(b_1,\ldots,b_{n-1})}{b_{n-1}-b_n}\\
&- \frac{1}{(n-2)!}\cdot\frac{Z_{n-1}(b_1,\ldots,b_{n-2},b_n)}{b_{n-1}-b_n},
\end{aligned}
$$

and we have the following recurrent formula:

$$
Z_n(b_1,\ldots,b_n) = (n-1)\cdot\frac{Z_{n-1}(b_1,\ldots,b_{n-2},b_{n-1}) - Z_{n-1}(b_1,\ldots,b_{n-2},b_n)}{b_{n-1}-b_n} \tag{A.1}
$$

From expression (12) for Z_n

$$
Z_n(b_1,\ldots,b_n) = -(n-1)!\cdot\sum_{k=1}^{n}\frac{\mathrm{e}^{-b_k}}{\prod_{\substack{j=1\\ j\neq k}}^{n}(b_k-b_j)},
$$

we infer

$$
Z_{n-1}(b_1,\ldots,b_{n-2},b_{n-1}) = -(n-2)!\cdot\sum_{k=1}^{n-1}\frac{\mathrm{e}^{-b_k}}{\prod_{\substack{j=1\\ j\neq k}}^{n-1}(b_k-b_j)}
$$

$$
Z_{n-1}(b_1,\ldots,b_{n-2},b_n) = -(n-2)!\cdot\sum_{\substack{k=1\\ k\neq n-1}}^{n}\frac{\mathrm{e}^{-b_k}}{\prod_{\substack{j=1\\ j\neq k,n-1}}^{n}(b_k-b_j)},
$$

which equivalently means

$$
-\frac{1}{(n-2)!}\cdot Z_{n-1}(b_1,\ldots,b_{n-2},b_{n-1}) = \sum_{k=1}^{n-2}\frac{\mathrm{e}^{-b_k}}{(b_k-b_{n-1})\prod_{\substack{j=1\\ j\neq k}}^{n-2}(b_k-b_j)} + \frac{\mathrm{e}^{-b_{n-1}}}{\prod_{j=1}^{n-2}(b_{n-1}-b_j)}
$$

$$
-\frac{1}{(n-2)!}\cdot Z_{n-1}(b_1,\ldots,b_{n-2},b_n) = \sum_{k=1}^{n-2}\frac{\mathrm{e}^{-b_k}}{(b_k-b_n)\prod_{\substack{j=1\\ j\neq k}}^{n-2}(b_k-b_j)} + \frac{\mathrm{e}^{-b_n}}{\prod_{j=1}^{n-2}(b_n-b_j)}
$$

form the difference

$$
\begin{aligned}
&-\frac{1}{(n-2)!}\cdot(Z_{n-1}(b_1,\ldots,b_{n-2},b_{n-1}) - Z_{n-1}(b_1,\ldots,b_{n-2},b_n))\\
&\quad= \sum_{k=1}^{n-2}\left[\frac{\mathrm{e}^{-b_k}}{(b_k-b_{n-1})\prod_{\substack{j=1\\ j\neq k}}^{n-2}(b_k-b_j)} - \frac{\mathrm{e}^{-b_k}}{(b_k-b_n)\prod_{\substack{j=1\\ j\neq k}}^{n-2}(b_k-b_j)}\right]\\
&\quad+ \frac{\mathrm{e}^{-b_{n-1}}}{\prod_{j=1}^{n-2}(b_{n-1}-b_j)} - \frac{\mathrm{e}^{-b_n}}{\prod_{j=1}^{n-2}(b_n-b_j)}
\end{aligned}
$$

$$= \sum_{k=1}^{n-2} \left[\left(\frac{1}{b_k - b_{n-1}} - \frac{1}{b_k - b_n} \right) \frac{e^{-b_k}}{\prod_{\substack{j=1 \\ j\neq k}}^{n-2} (b_k - b_j)} \right] + \frac{e^{-b_{n-1}}}{\prod_{j=1}^{n-2} (b_{n-1} - b_j)} - \frac{e^{-b_n}}{\prod_{j=1}^{n-2} (b_n - b_j)}. \tag{A.2}$$

Perform interim calculation

$$\frac{1}{b_k - b_{n-1}} - \frac{1}{b_k - b_n} = \frac{b_k - b_n - b_k + b_{n-1}}{(b_k - b_{n-1})(b_k - b_n)} = \frac{b_{n-1} - b_n}{(b_k - b_{n-1})(b_k - b_n)}.$$

Then the difference (A.2) equals

$$(b_{n-1} - b_n) \cdot \sum_{k=1}^{n-2} \frac{e^{-b_k}}{(b_k - b_{n-1})(b_k - b_n) \prod_{\substack{j=1 \\ j\neq k}}^{n-2} (b_k - b_j)} + \frac{e^{-b_{n-1}}}{\prod_{j=1}^{n-2} (b_{n-1} - b_j)} - \frac{e^{-b_n}}{\prod_{j=1}^{n-2} (b_n - b_j)}$$

$$= (b_{n-1} - b_n) \cdot \sum_{k=1}^{n-2} \frac{e^{-b_k}}{\prod_{\substack{j=1 \\ j\neq k}}^{n} (b_k - b_j)} + \frac{e^{-b_{n-1}}}{\prod_{j=1}^{n-2} (b_{n-1} - b_j)} - \frac{e^{-b_n}}{\prod_{j=1}^{n-2} (b_n - b_j)}.$$

Divide it into $b_{n-1} - b_n$:

$$-\frac{1}{(n-2)!} \cdot \frac{Z_{n-1}(b_1, \ldots, b_{n-2}, b_{n-1}) - Z_{n-1}(b_1, \ldots, b_{n-2}, b_n)}{b_{n-1} - b_n}$$

$$= \sum_{k=1}^{n-2} \frac{e^{-b_k}}{\prod_{\substack{j=1 \\ j\neq k}}^{n} (b_k - b_j)} + \frac{e^{-b_{n-1}}}{(b_{n-1} - b_n) \prod_{j=1}^{n-2} (b_{n-1} - b_j)} - \frac{e^{-b_n}}{(b_{n-1} - b_n) \prod_{j=1}^{n-2} (b_n - b_j)}$$

$$= \sum_{k=1}^{n-2} \frac{e^{-b_k}}{\prod_{\substack{j=1 \\ j\neq k}}^{n} (b_k - b_j)} + \frac{e^{-b_{n-1}}}{\prod_{\substack{j=1 \\ j\neq n-1}}^{n} (b_{n-1} - b_j)} + \frac{e^{-b_n}}{\prod_{\substack{j=1 \\ j\neq n}}^{n} (b_n - b_j)}$$

$$= \sum_{k=1}^{n} \frac{e^{-b_k}}{\prod_{\substack{j=1 \\ j\neq k}}^{n} (b_k - b_j)}$$

$$= -\frac{1}{(n-1)!} Z_n(b_1, \ldots, b_n).$$

Therefore

$$(n-1) \cdot \frac{Z_{n-1}(b_1, \ldots, b_{n-2}, b_{n-1}) - Z_{n-1}(b_1, \ldots, b_{n-2}, b_n)}{b_{n-1} - b_n} = Z_n(b_1, \ldots, b_n).$$

Formula (14) for the eigenvalues of the density operator ρ is obtained by routine calculation: taking partial derivatives of the expression for $Z_n(b_1, \ldots, b_n)$ over the variables b_s.

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